JOURNAL OF ALGEBRA **131**, 455–465 (1990)

On Lifting Generalized Characters in Finite Groups

HENRY S. LEONARD

Department of Mathematical Sciences, Northern Illinois University, DeKalb, Illinois 60115

Communicated by Leonard Scott

Received June 26, 1988

DEDICATED TO WALTER FEIT ON THE OCCASION OF HIS 60TH BIRTHDAY

1. INTRODUCTION

During the last twenty-five years there has been steady progress in the study of lifting operators σ taking certain generalized characters of a subgroup H to generalized characters of a finite group G. Often results about σ can be applied to yield theorems on the existence of a relative normal complement in G. (Given $H_0 \lhd H \leqslant G$ and given $G_0 \lhd G$, we call G_0 a *normal complement of H over* H_0 if $G = G_0 H$ and $H_0 = G_0 \cap H$, and hence $G/G_0 \cong H/H_0$.) Particularly important contributions in this direction were made by Brauer [1], Dade [2], and Robinson [6]. Earlier references for the theory and applications of lifting operators are given by Robinson [6] and in the papers he cites.

In Section 2 we define a lifting operator σ so that, under quite general conditions, certain complex valued class functions θ of a subgroup H are lifted to complex valued class functions θ^σ of G. Lemma 2.1 tells us that if θ is a rational combination of irreducible characters of H then θ^σ is a rational combination of irreducible characters of G. Section 3 contains three results which assert that, under certain conditions, if θ is a generalized character of H, then θ^σ is a generalized character of G. Two of these lifting theorems are applied in Section 4 to obtain theorems about the existence of relative normal complements.

The objects of this paper are to obtain new lifting theorems and to apply them. The principal application made here is the derivation of Theorem 4.3 on relative normal complements, due to Brauer [1] and Dade [2]. It is desirable to obtain lifting theorems which yield normal complement theorems and have other applications as well. I hope this work will lead to other applications. In Theorem 3.3 Assumption 2(i) is weaker than the

0021-8693/90 $3.00

corresponding assumption in Corollary 3.2. It would be desirable to weaken or remove assumption 2(ii) in Theorem 3.3, but this seems to be difficult.

2. Notation and Preliminaries

Throughout the paper G will denote a finite group. We write $x \sim y$ to mean that x and y are conjugate elements of G. Let π be a set of primes. The complementary set of primes will be denoted by π'. We say that G is a *π-group* if the order $|G|$ of G is divisible only by primes in π. An element x is a *π-element* if the subgroup $\langle x \rangle$ is a π-group. Every element x of G has a unique decomposition $x = x_\pi x_{\pi'} = x_{\pi'} x_\pi$ into a π-element x_π and a π'-element $x_{\pi'}$. Two elements x and y are said to belong to the same *π-section* of G if their π-parts x_π and y_π are G-conjugate. We denote by $S_\pi^G(x)$ the π-section of G containing the element x. If S is a subset of G we let $S^{G,\pi}$ denote the union of all π-sections of G that intersect S.

Now we define an operator σ lifting certain class functions θ with domain H to functions θ^σ with domain G.

Definition. *Let G be a finite group, let $H \leqslant G$, let π be a set of primes, and let A be some union of π-sections of H. Let θ be a complex valued function with domain H such that*

(i) *θ vanishes on $H \backslash A$ and*

(ii) *for each π-element a of A, θ is constant on $S_\pi^G(a) \cap A$.*

Define θ^σ with domain G so that $\theta^\sigma | A = \theta | A$, θ^σ vanishes on all π-sections of G not meeting A, and θ^σ is constant on each π-section of G.

Throughout the paper σ will have this meaning. When it is known that for each π-element a of A, θ is constant on $S_\pi^G(a) \cap H$, then it follows that $\theta^\sigma | H = \theta$. Our first result is related to a lemma of Robinson [6, Lemma 4].

Lemma 2.1. *Suppose G, H, A, and θ satisfy the conditions of the definition of σ. Suppose θ is a rational combination of characters of H. Then θ^σ is a rational combination of irreducible characters of G.*

To prove Lemma 2.1 we apply a general lemma of Gluck and Isaacs. Let G be a finite group and let ξ be a primitive $|G|$th root of unity. Let $\mathbb{C}$ denote the field of complex numbers. For each $\tau \in \mathrm{Aut}(\mathbb{C})$, $\xi^\tau = \xi^{m(\tau)}$ for some integer $m(\tau)$. Then we have

Lemma 2.2 (Gluck and Isaacs [3, Lemma 3.2]). *Let θ be a complex valued class function on a finite group G. Then θ is a rational combination* *of characters of* G *if and only if, for all* $g \in G$ *and all* $\tau \in \operatorname{Aut}(\mathbb{C})$, $\theta(g)^\tau = \theta(g^{m(\tau)})$.

Proof of Lemma 2.1. Let $\tau \in \operatorname{Aut}(\mathbb{C})$ and let ξ and $m(\tau)$ be defined as above. Now

$$\theta^\sigma(g^{m(\tau)}) = \theta^\sigma(g_\pi^{m(\tau)})$$

for all $g \in G$.

Case 1. *Suppose* $g \in G$ *and suppose* g_π *is conjugate to an element of* A. Without loss of generality we may assume $g_\pi \in A$.

Case 1a. $S_\pi^G(g_\pi^{m(\tau)})$ *meets* A. Let a be a π-element in the intersection. Now $a = b^{m(\tau)}$ for some power b of a, because $m(\tau)$ is a π'-number. Furthermore $g_\pi^{m(\tau)} \sim b^{m(\tau)}$, and taking suitable powers we have $g_\pi \sim b$. By definition of σ, $\theta^\sigma(g_\pi^{m(\tau)}) = \theta(b^{m(\tau)})$. Applying Lemma 2.2 to θ we have $\theta(b^{m(\tau)}) = \theta(b)^\tau$. On the other hand

$$\theta^\sigma(g)^\tau = \theta^\sigma(g_\pi)^\tau = \theta(g_\pi)^\tau. \tag{1}$$

If $b \in A$ then $\theta(b) = \theta(g_\pi)$, so that $\theta^\sigma(g^{m(\tau)}) = \theta^\sigma(g)^\tau$.

Now suppose $b \notin A$. If $g_\pi^{m(\tau)} \in A$ then $\theta(g_\pi)^\tau = \theta(g_\pi^{m(\tau)}) = \theta^\sigma(g_\pi^{m(\tau)})$. If $g_\pi^{m(\tau)} \notin A$ then

$$\theta(g_\pi)^\tau = \theta(g_\pi^{m(\tau)}) = 0 = \theta(b)^\tau = \theta^\sigma(g_\pi^{m(\tau)}),$$

so that again, in both these cases, Eq. (1) implies $\theta^\sigma(g^{m(\tau)}) = \theta^\sigma(g)^\tau$.

Case 1(b). $S_\pi^G(g_\pi^{m(\tau)})$ *does not meet* A. Then $\theta^\sigma(g_\pi^{m(\tau)}) = 0 = \theta(g_\pi^{m(\tau)}) = \theta(g_\pi)^\tau$, since θ vanishes on $H \backslash A$. Thus by Eq. (1) $\theta^\sigma(g^{m(\tau)}) = \theta^\sigma(g)^\tau$ in this case.

Case 2. *Suppose* $g \in G$ *and suppose* g_π *has no conjugates in* A. First we proceed just as in Case 1a, but now we get $\theta(b)^\tau = 0 = \theta^\sigma(g_\pi)^\tau$, because $b \in H \backslash A$ and by the assumption of Case 2. Finally we have the case that neither g_π nor $g_\pi^{m(\tau)}$ has a conjugate in A. Then

$$\theta^\sigma(g^{m(\tau)}) = 0 = \theta^\sigma(g) = \theta^\sigma(g)^\tau.$$

Thus for all $g \in G$ we have $\theta^\sigma(g)^\tau = \theta^\sigma(g^{m(\tau)})$, and by Lemma 2.2 θ^σ satisfies the conclusion of Lemma A.

In the next section we shall apply Lemma 2.1 to obtain three results which provide conditions sufficient for θ^σ to be a generalized character of G when θ is assumed to be a generalized character of H. Because of Lemma 2.1 it will suffice in Section 3 to prove that θ^σ is an algebraic integer combination of irreducible characters of G. For any group G we denote by $\mathrm{Alg}(G)$ the set of all algebraic integer combinations of irreducible characters of G. We state two lemmas of G. R. Robinson, which will be needed in Section 3.

LEMMA 2.3 [6, Lemma 2 and Remark, p. 436]. *Let H be a finite group and let τ be a set of primes.*

(i) *If β is a complex valued class function of H which is constant on τ-sections, and if $\beta \mid T \in \mathrm{Alg}(T)$ for every nilpotent τ-subgroup T of H, then $\beta \in \mathrm{Alg}(H)$.*

(ii) *If $\varphi \in \mathrm{Alg}(H)$ and if $\varphi^*(h) = \varphi(h_\tau)$ for each $h \in H$, then $\varphi^* \in \mathrm{Alg}(H)$.*

LEMMA 2.4 [6, Lemma 3(ii)]. *Let H be a finite group and let τ be a set of primes. Let x be a τ-element of H, and let β be defined on H by*

$$\beta(h) = \begin{cases} |C_H(x)|_\tau & \text{on } S_\tau^H(x), \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\beta \in \mathrm{Alg}(H)$.

3. LIFTING GENERALIZED CHARACTERS

THEOREM 3.1. *Let G be a finite group, let H be a subgroup of G, and let A be a subset of H. Let π be a set of primes. Let θ be a generalized character of H. Assume the following conditions are satisfied.*

1. *The set A is a union of π-sections of H.*
2. *For each π-element a in A the number $n(a)$ of left cosets gH of H in G which are fixed under the action $gH \mapsto agH$ is a π'-number.*
3. *The generalized character θ vanishes on $H \backslash A$.*
4. *For each π-element a of A, θ is constant on $S_\pi^G(a) \cap H$.*

Then θ^σ is a generalized character of G.

Proof. The function θ^σ was defined in Section 2. According to Lemma 2.1, θ^σ is a rational combination of characters of G. Therefore it will suffice here to prove that $\theta^\sigma \in \mathrm{Alg}(G)$.

We may put

$$A = \bigcup_{i=1}^{s} (S_\pi^G(a_i) \cap A) \qquad \text{(disjoint)},$$

where each a_i is a π-element of A. For each i, define

$$\varphi_i(g) = \begin{cases} |G|_\pi \theta(a_i) & \text{if } \ g \in S_\pi^G(a_i), \\ 0 & \text{otherwise.} \end{cases}$$

Then by Lemma 2.4, $\varphi_i \in \mathrm{Alg}(G)$ for each i.

For each i, θ is constant on $S_\pi^G(a_i) \cap H$. Therefore for the induced character θ^G we have $\theta^G(a_i) = n(a_i)\, \theta(a_i)$, because for $g \in G$, a_i^g contributes to $\theta^G(a_i)$ if and only if $a_i^g \in H$. Define θ^* to agree with θ^G on all π-elements of G and to be constant on all π-sections of G. Then by Lemma 2.3(ii), $\theta^* \in \mathrm{Alg}(G)$. Clearly θ^* is constant on $S_\pi^G(a) \cap H$ for each π-element a of A, and $\theta^* \mid H$ vanishes on $H \backslash A$.

Define a sequence of class functions on H as

$$\theta_1 = \theta, \qquad \theta_{k+1} = (\theta_k)^* \mid H.$$

Then $(\theta_k)^* \in \mathrm{Alg}(G)$ for each k. Also for each i and each k we have $(\theta_k)^*(a_i) = n(a_i)^k \theta(a_i)$. If m is the number of positive integers less than $|G|_\pi$ and prime to $|G|_\pi$ then by assumption 2 we have

$$n(a_i)^m \equiv 1 \pmod{|G|_\pi}$$

for each i.

By definition of $(\theta_k)^*$ we have

$$((\theta_m)^* - \theta^\sigma)(a_i) = [n(a_i)^m - 1]\, \theta(a_i)$$

for each i. Therefore $(\theta_m)^* - \theta^\sigma \in \mathrm{Alg}(G)$, since $\varphi_i \in \mathrm{Alg}(G)$ for each i. It follows that $\theta^\sigma \in \mathrm{Alg}(G)$, because $(\theta_m)^* \in \mathrm{Alg}(G)$, and the proof is complete.

Remark. Under the assumptions of Theorem 3.1, if $p \in \pi$ and if there exists a p-element a in A, then the total number $[G:H]$ of left cosets of H is not divisible by p.

COROLLARY 3.2. *Let G be a finite group, let H be a subgroup of G, and let A be a subset of H. Let π be a set of primes. Let θ be a generalized character of H. Assume the following conditions are satisfied.*

1. *The set A is a union of π-sections of H.*
2. *Whenever two π-elements of A are G-conjugate they are H-conjugate.*
3. *No π-element of A is G-conjugate to an element of $H \backslash A$.*
4. *For each π-element a of A, $[C_G(a) : C_H(a)]$ is a π'-number.*
5. *The generalized character θ vanishes on $H \backslash A$ and is constant on all π-sections of H.*

Then θ^σ is a generalized character of G.

Proof. We verify that this corollary is a special case of Theorem 3.1. Let a be a π-element of A. For $g \in G$ the left coset gH is fixed by a if and only if $g \in C_G(a)H$, by Assumptions 3 and 2. Therefore the number $n(a)$ of left cosets gH fixed by a is $[C_G(a):C_H(a)]$. Thus Assumption 4 shows that Condition 2 of Theorem 3.1 is satisfied. Clearly Condition 4 of Theorem 3.1 follows from Assumptions 2, 3, and 5 of the corollary. Hence the proof of Corollary 3.2 is complete.

THEOREM 3.3. *Let G be a finite group, let H be a subgroup of G, and let A be a subset of H. Let π be a set of primes. Assume*

1. *The index $[G:H]$ is a π'-number.*
2. *The set A is a union of π-sections of H such that*

(i) *For every π-element x of A, whenever $p \in \pi$, p is not a divisor of $|\langle x\rangle|$, and P is a p-subgroup of $C_G(x)$, then the subgroup $\langle x\rangle P$ is G-conjugate to a subgroup of H.*

(ii) *Whenever $g \in G$ and $g_p \sim a \in A$ for some a then $g_\pi \sim b \in H$ for some b.*

(iii) *No π-element of A is G-conjugate to an element of $H \setminus A$.*

3. *Let θ be a generalized character of H such that*

(i) *θ vanishes on $H \setminus A$,*

(ii) *For each π-element $a \in A$, θ is constant on $S_\pi^G(a) \cap A$, and*

(iii) *For every π-element h of H, if $p \in \pi$ and $\tau = \pi \setminus \{p\}$ and $h_\tau \in H \setminus A$ then $\theta(h) = \theta(h_p)$.*

Then θ^σ is a generalized character of G.

Proof. According to Lemma 2.1, it will suffice to prove that $\theta^\sigma \in \mathrm{Alg}(G)$. If $|\pi| = 1$ then θ and θ^σ are constant on p-sections, where $\pi = \{p\}$. If P is a Sylow p-subgroup of G, then by Assumption 1 there is a conjugate Q of P with $Q \leqslant H$. Then $\theta^\sigma | Q = \theta | Q$, so according to Lemma 2.3(i) $\theta^\sigma \in \mathrm{Alg}(G)$. Therefore we may assume $|\pi| > 1$. We give the proof in a series of steps.

Step 1. *It suffices to prove that if $p \in \pi$ and $\tau = \pi \setminus \{p\}$ then $|G|_\tau \theta^\sigma \in \mathrm{Alg}(G)$.*

Proof. Let $\pi = \{p_1, p_2, ..., p_n\}$ and let $\tau_i = \pi \setminus \{p_i\}$ for each i. We have

$$1 = \sum_{i=1}^{n} s_i\, |G|_{\tau_i}$$

for some integers s_i, so

$$\theta^\sigma = \sum_{i=1}^{n} s_i |G|_{\tau_i} \theta^\sigma.$$

The proof of Step 1 is complete.

The set $A^{G,\pi}$ is contained in a minimal union of disjoint τ-sections $S_\tau^G(x)$ of G. Here x is a τ-element of G and possibly $x=1$. After replacing x by a conjugate if necessary, we may assume $x \in H$. Consider the situation when $x \in A$. According to Assumptions 2(i) and (iii) we may assume, after replacing x by a G-conjugate if necessary, that $C_H(x)$ contains a Sylow p-subgroup of $C_G(x)$.

Define

$$A_x = A \cap S_\tau^H(x), \qquad \text{where} \quad x \in A.$$

We claim that

$$A_x^{G,\pi} = A^{G,\pi} \cap S_\tau^G(x). \tag{2}$$

It is clear that the first set is a subset of the second. Now suppose $y \in A^{G,\pi} \cap S_\tau^G(x)$. Then $y_\tau \sim x$, and hence $y_\pi \sim xv$ for some p-element v of $C_G(x)$. For some $g \in C_G(x)$, $v^g = u \in C_H(x)$, and hence $y_\pi \sim xu \in C_H(x)$. But also $y_\pi \sim a \in A$ for some a, and $a \sim xu$. It follows from Assumption 2(iii) that $xu \in A$. Thus $xu \in A \cap S_\tau^H(x)$, so $y \in A_x^{G,\pi}$. This completes the proof of Eq. (2).

We may put

$$A_x = \bigcup_{i=1}^{m} S_\pi^H(xy_i),$$

where each y_i is a p-element of $C_H(x)$, and the π-sections $S_\pi^H(xy_i)$ are distinct and are contained in A. Define ψ on H by

$$\psi(h) = \begin{cases} \theta(h) & \text{on } A_x, \\ 0 & \text{elsewhere.} \end{cases}$$

By Lemma 2.4, $|G|_\tau \psi$ is a product of algebraic integer combinations of characters of H.

Now define

$$A_0 = \{a \in A : a_\tau \in H \backslash A\}.$$

The above discussion, together with Assumption 2(iii), implies that $A^{G,\pi}$ is the disjoint union of $A_0^{G,\pi}$ and selected sets $A_x^{G,\pi}$, where $x \in A$. Define φ on H by

$$\varphi(h) = \begin{cases} \theta(h) & \text{on } A_0, \\ 0 & \text{elsewhere.} \end{cases}$$

We form ψ^σ and φ^σ, where σ is defined in terms of A_x and A_0, respectively, instead of A. Then θ^σ is a sum of φ^σ and of selected functions ψ^σ as x varies. This proves

Step 2. *It suffices to prove that* $|G|_\tau\psi^\sigma \in \mathrm{Alg}(G)$ *and* $|G|_\tau\varphi^\sigma \in \mathrm{Alg}(G)$.

Define

$$B_x = \bigcup_{i=1}^{m} S_\pi^{C_H(x)}(xy_i).$$

Define β on $C_H(x)$ by

$$\beta(h) = \begin{cases} |G|_\tau\theta(h) & \text{on } B_x, \\ 0 & \text{elsewhere.} \end{cases}$$

Now $B_x = A \cap S_\tau^{C_H(x)}(x)$. Furthermore $|G|_\tau\psi = \beta^H$. By Lemma 2.4 β is a product of algebraic integer combinations of characters of $C_H(x)$, since θ vanishes outside A.

In the definition of σ in Section 2 we let $C_G(x)$ take the part of G, $C_H(x)$ that of H, B_x that of A, and β that of θ. Then β^σ is defined on $C_G(x)$. Furthermore $|G|_\tau\psi^\sigma = (\beta^H)^\sigma$.

Step 3. $|G|_\tau\psi^\sigma = (\beta^H)^\sigma = (\beta^\sigma)^G$.

Proof. For $g \in G$ we have

$$(\beta^H)^\sigma(g) = \begin{cases} \beta(h) & \text{if } \; g_\pi \sim h \in B_x, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand

$$(\beta^\sigma)^G(g) = \begin{cases} \beta^\sigma(y) & \text{if } \; g \sim y \in A^{C_G(x),\pi} \cap S_\tau^{C_G(x)}(x), \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the definition of β^σ that $(\beta^H)^\sigma = (\beta^\sigma)^G$. This gives us

Step 4. *If* $\beta^\sigma \in \mathrm{Alg}(C_G(x))$ *then* $|G|_\tau\psi^\sigma \in \mathrm{Alg}(G)$.

Step 5. *We have* $\beta^\sigma \in \mathrm{Alg}(C_G(x))$.

Proof. We apply Theorem 3.1 to $C_G(x)$, $C_H(x)$, B_x, and β. Clearly Assumptions 1, 3, and 4 of Theorem 3.1 are satisfied. By the way x was chosen, $[C_G(x): C_H(x)]$ is a p'-number. It follows that the number $n(xy_i)$ of cosets $gC_H(x)$ in $C_G(x)$ fixed by y_i is prime to p, because y_i is a p-element so its non-trivial orbits have length divisible by p. The same cosets are fixed by xy_i. To see that this weaker condition is sufficient, we follow the proof of Theorem 3.1. We form the sequence $\{\beta_k\}$ of members of $\mathrm{Alg}(C_H(x))$, and we find that

$$((\beta_m)^* - \beta^\sigma)(xy_i) = [n(xy_i)^m - 1]\, \beta(xy_i),$$

where $n(xy_i)^m \equiv 1 \pmod{|C_G(x)|_p}$. Since $\beta(xy_i) = |G|_\tau \theta(xy_i)$, we find that $(\beta_m)^* - \beta^\sigma \in \mathrm{Alg}(C_G(x))$ and $\beta^\sigma \in \mathrm{Alg}(C_G(x))$. This completes the proof of Step 5.

Now we define γ on H by $\gamma(h) = \theta(h_p)$. By Lemma 2.3(ii), $\gamma \in \mathrm{Alg}(H)$. Define $\gamma^{\sigma'}$ on G by

$$\gamma^{\sigma'}(g) = \gamma(h) \qquad \text{if} \quad g_p \sim h \in H.$$

Clearly $\gamma^{\sigma'}$ is well defined, by Assumption 3(ii).

Step 6. *We have* $\gamma^{\sigma'} \in \mathrm{Alg}(G)$.

Proof. If P is a Sylow p-subgroup of G then by Assumption 1 there is a conjugate Q of P with $Q \leqslant H$. Furthermore $\gamma^{\sigma'} \,|Q = \gamma|\, Q$, so according to Lemma 2.3(i) $\gamma^{\sigma'} \in \mathrm{Alg}(G)$.

Step 7. *We have* $|G|_\tau \varphi^\sigma \in \mathrm{Alg}(G)$.

Proof. If $g \in G$ and $g_\pi \sim h \in A_0$, then $\varphi^\sigma(g) = \varphi(h) = \theta(h)$ and $\gamma^{\sigma'}(g) = \gamma(h_p) = \theta(h_p)$. By definition of A_0, $h_\tau \in H \backslash A$. Therefore Assumption 3(iii) implies that $\theta(h) = \theta(h_p)$. It follows that $\varphi^\sigma(g) = \gamma^{\sigma'}(g)$ if $g \in A_0^{G,\pi}$. Thus for all $g \in G$

$$\varphi^\sigma(g) = \begin{cases} \gamma^{\sigma'}(g) & \text{if} \quad g \in A_0^{G,\pi}, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that if $g \in A_0^{G,\tau} \backslash A_0^{G,\pi}$ then $\gamma^{\sigma'}(g) = 0$. Suppose $g \in A_0^{G,\tau} \backslash A_0^{G,\pi}$. Then $g_p \neq 1$. If g_p has a conjugate in A, then, by Assumption 2(ii), $g_\pi \sim b \in H$ for some b. Then $b_\tau \in H \backslash A$ since $g_\tau \sim c \in H \backslash A$ for some c. By Assumption 3(iii), $\theta(b) = \theta(b_p)$. But $g \notin A_0^{G,\pi}$, so $b \in H \backslash A$, $\theta(b) = 0$, and $\gamma^{\sigma'}(g) = 0$. Hence we may assume $g_p \sim h \in H \backslash A$ for some h. Then $\theta(h) = 0$ and $\gamma^{\sigma'}(g) = 0$ in this case also.

Now for all $g \in G$ we have

$$\varphi^\sigma(g) = \begin{cases} \gamma^{\sigma'}(g) & \text{if} \quad g \in A_0^{G,\tau}, \\ 0 & \text{otherwise.} \end{cases}$$

We multiply this equation by $|G|_\tau$ and apply Lemma 2.4. It follows that $|G|_\tau \varphi^\sigma$ is a product of algebraic integer combinations of characters of G, and the proof of Step 7 is complete.

When we combine Steps 2, 4, 5, and 7, we see that the proof of the theorem is complete.

4. Relative Normal Complements

In order to apply the conclusions of Section 3 to obtain results about relative normal complements, we use the following theorem, which was proved by Dade [2] and was recast by the author [4, Theorem 4.2].

THEOREM 4.1. *Let G be a finite group, let H be a subgroup of G, and $H_0 \lhd H$. Let A denote $H \backslash H_0$. Let π be the set of prime divisors of $[H:H_0]$. Assume*

1. *Whenever two π-elements a and b of A are G-conjugate then aH_0 and bH_0 are conjugate in H/H_0.*

2. *Whenever θ is a generalized character of H which vanishes on H_0 and whose kernel contains H_0, then θ^σ is a generalized character of G.*

Then there exists a unique normal complement G_0 of H over H_0, and

$$G_0 = G \backslash A^{G,\pi}.$$

Here the condition that θ be a generalized character of H whose kernel contains H_0 means that θ is an integer combination of irreducible characters of H whose kernels contain H_0.

Now we combine Theorem 3.1 and Theorem 4.1 to obtain the following result.

COROLLARY 4.2. *Let G be a finite group, H a subgroup of G, and $H_0 \lhd H$. Let A denote $H \backslash H_0$. Let π be the set of prime divisors of $[H:H_0]$. Assume*

1. *Whenever two π-elements a and b of H are G-conjugate then aH_0 and bH_0 are conjugate in H/H_0.*

2. *For each π-element a in A the number $n(a)$ of left cosets gH of H in G which are fixed under the action $gH \mapsto agH$ is a π'-number.*

Then there exists a unique normal complement G_0 of H over H_0, and $G_0 = G \backslash A^{G,\pi}$.

In Corollary 4.2 Assumption 1 concerns all π-elements a and b in H, in contrast to Assumption 1 of Theorem 4.1, because we need to be sure that Assumption 4 of Theorem 3.1 will be satisfied.

Remarks. The remark following the proof of Theorem 3.1 can be strengthened here. If the assumptions of Theorem 4.1 hold then there do exist p-elements in A for every prime p in π. Therefore the hypotheses of Corollary 4.2 imply that $[G:H]$ is a π'-number. Corollary 4.2 is related to the Brauer–Suzuki Theorem [1, Theorem 3], [5], as Corollary 3.2 suggests.

We could combine Corollary 3.2 and Theorem 4.1 to obtain another corollary about relative normal complements. But again our hypotheses would imply that $[G:H]$ is a π'-number. For this reason the result would be weaker than the result of combining Theorem 3.3 and Theorem 4.1. Therefore we omit the first of these, and we proceed now to the second.

All the hypotheses of Theorem 3.3 are easily seen to be consequences of the assumptions of Theorem 4.3 stated below. (As regards Hypothesis 2(ii) of Theorem 3.3, it is a consequence of a lemma of Brauer [1, Lemma 2].) Therefore when we combine Theorem 3.3 and Theorem 4.1, we have

THEOREM 4.3. *Let G be a finite group, H a subgroup of G, and $H_0 \lhd H$. Let A denote $H \backslash H_0$. Let π be the set of prime divisors of $[H:H_0]$. Assume*

1. *Whenever two π-elements a and b of H are G-conjugate then aH_0 and bH_0 are conjugate in H/H_0.*
2. *The index $[G:H]$ is a π'-number.*
3. *For every π-element a of A, whenever $p \in \pi$, and p is not a divisor of $|\langle a \rangle|$, and P is a p-subgroup of $C_G(a)$, then the subgroup $\langle a \rangle P$ is G-conjugate to a subgroup of H.*

Then there exists a unique normal complement G_0 of H over H_0, and $G_0 = G \backslash A^{G,\pi}$.

Theorem 4.3 was proved by Dade [2, Theorem 2], and it had been proved earlier by Brauer under stronger hypotheses [1, Theorem 1].

ACKNOWLEDGMENTS

This research was done while I was a Visitor at the University of Manchester Institute of Science and Technology, on sabbatical leave from Northern Illinois University. I thank both these universities. I am grateful to Dr. G. R. Robinson for several very helpful conversations.

REFERENCES

1. R. BRAUER, On quotient groups of finite groups. *Math. Z.* **83** (1964), 72–84.
2. E. C. DADE, On normal complements to sections of finite groups, *J. Austral. Math. Soc. Ser. A* **19** (1975), 257–262.
3. D. GLUCK AND I. M. ISAACS, Tensor induction of generalized characters and permutation characters, *Illinois J. Math.* **27** (1983), 514–518.
4. H. S. LEONARD, On relative normal complements in finite groups, *Arch. Math. (Basel)* **40** (1983), 97–108.
5. H. S. LEONARD, On the Brauer–Suzuki theorem, *in* "The Arcata Conference on Representations of Finite Groups," pp. 57–64, Proceedings of Symposia in Pure Mathematics, Vol. 47, *Amer. Math. Soc.* Providence, RI, 1987.
6. G. R. ROBINSON, Blocks, isometries, and sets of primes, *Proc. London Math. Soc. (3)* **51** (1985), 432–448.